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Approximation of Fixed Points of Multivalued ρ-nonexpansive Mappings for Ishikwa Iterative Scheme in Modular Function Spaces

Anju Panwar[1] and Reena[2]

[1]Department of Mathematics, Maharshi Dayanand University, Rohtak (Haryana) - 124001, India
[2]Department of Mathematics, Maharshi Dayanand University, Rohtak (Haryana) - 124001, India
*E-mail: [1]anjupanwar15@gmail.com, [2]morwal80@gmail.com**

Abstract—*The aim of this research paper is to introduce Ishikwa iterative scheme to approximate the fixed points of multivalued $\rho-$ nonexpansive mappings in modular function spaces.*
Keywords: Common fixed point, multivalued $\rho-$nonexpansive, Noor-type iteration scheme, modular function space.

1. INTRODUCTION

The concept of order spaces was generalized by Nakano [13] to the modular spaces in 1950. Then it was further generalized and redefined by Musielak and Orlicz [12]. As the modular function spaces are the generalization of some class of Banach spaces, so many analysts show their interest in working in this field in modular function spaces. Khamsi, Kozlowski and Reich [7] were the first who initiated the study of fixed point theory in these spaces in 1990. On the basis of their results, many work has been done in these spaces. For instance, in 2006, Dhomopongsa [3] proved the existence of fixed point for $\rho-$contraction mapping. Kozlowski [2, 6, 7, 9] has contributed a lot in this field with on his own and with his collaborators.

Until 2012, there was no result for the approximation of fixed point in modular function. Then Dehaish and Kozlowski [2] were the first who try to fill this gap by using Mann iteration for asymptotically pointwise nonexpansive mappings in 2012. Abdou *et.al.* [1] introduced the concept of approximation of common fixed points of two $\rho-$nonexpansive by using Ishikawa iteration procedure in modular function spaces in 2014. However all the above work was done for single valued mappings. The existence and approximation of fixed points through the well known iterative schemes of Mann [11], and Ishikawa [4] of multivalued mappings drew the attention of many mathematicians like Kozlowski, Latif, Kutib, Khan, Abbas etc. The reason of their interest may be due to its applications in real world problems such as Game theory, Market Economy etc. and other applied mathematics such as nonlinear optimization and differential equations.

In 2014, Khan and Abbas [8] were the first who gave the approximation theorem for fixed points of a multivalued $\rho-$ nonexpansive mapping in these spaces by using Mann iteration scheme. In this paper, we make an attempt to approximate the fixed points of $\rho-$nonexpansive multivalued mappings in these spaces for Ishikwa iteration scheme. Our results will extend, generalize and improve many existing results in literature.

2. BASIC DEFINITIONS

Let Ω be a nonempty set and Σ be a nontrivial $\sigma-$algebra of subsets of Ω. Let Ƥ be a nontrivial δ-ring of subsets of Ω which means that Ƥ is closed under countable intersection, and finite union and differences. Suppose that E∩A ∈ Ƥ for any E ∈ Ƥ and A ∈ Σ. Let us assume that there exists an increasing sequence of sets K_n ∈ Ƥ such that Ω = ∪K_n. By Ɛ we denote the linear space all simple functions with support from Ƥ. Also $\mathcal{M}_\infty$ denotes the space of all extended measurable functions, i.e., all functions f: Ω → [-∞, ∞] such that there exists a sequence

$\{g_n\} \subset$ Ɛ, $|g_n| \le |f|$ and $g_n(w) \to f(w)$

for all w ∈ Ω. We define

$\mathcal{M} = \{ f \in \mathcal{M}_\infty : |f(w)| < \infty \ \rho-a.e.\}$

Definition 2.1 [9] Let X be a vector space (R or C). A functional $\rho : X \to [0,\infty]$ is called a modular if for arbitrary f and g, elements of X, there hold the following:

i) $\rho(f) = 0 \Leftrightarrow f = 0$
ii) $\rho(\alpha f) = \rho(f)$ whenever $|\alpha| = 1$
iii) $\rho(\alpha f + \beta g) \le \rho(f) + \rho(g)$
whenever $\alpha,\ \beta \ge 0,\ \alpha + \beta = 1$

If we replace (iii) by

iv) $\rho(\alpha f+\beta g)\le\alpha\,\rho(f)+\beta\rho(g)$

whenever $\alpha,\beta\ge 0,\ \alpha+\beta=1$

Then modular ρ is called convex.

Definition 2.2[9] If ρ is convex modular in X, then the set defined by

$$L_\rho=\{f\in\mathrm{M}: lim_{\lambda\to 0}\rho(\lambda f)=0\}$$

is called modular function space. Generally, the modular ρ is not subadditive and therefore does not behave as a norm or a distance. However, the modular space L_ρ can be equipped with an F- norm defined by

$$\|f\|_\rho=inf\{a>0:\rho\left(\frac{f}{a}\right)\le a\}.$$

In the case ρ is convex modular,

$$\|f\|_\rho=inf\{a>0:\rho\left(\frac{f}{a}\right)\le 1\}$$

defines a norm on modular space L_ρ and it is called Luxemburge norm.

Definition 2.3[9] Let $\rho:\mathrm{M}_\infty\to[0,\infty]$ be a nontrivial, convex and even function. Then ρ is a regular convex function pseudomodular if

(1) $\rho(0)=0$;

(2) ρ is monotone, i.e., $|f(w)|\le|g(w)|$ for any $w\,ò\,\Omega$ implies $\rho(f)\le\rho(g)$, where $f,g\in\mathrm{M}_\infty$;

(3) ρ is orthogonally subadditive, i.e., $\rho(f\chi_{A\cup B})\le\rho(f\chi_A)+\rho(f\chi_B)$ for any A, B ∈ ∑ such that $A\cap B\ne\phi,\ f\in\mathrm{M}_\infty$;

(4) ρ has Fatou property, i.e., $|f_n(w)|\uparrow|f(w)|$ for $w\,ò\,\Omega$ implies $\rho(f_n)\uparrow\rho(f)$, where $f\in\mathrm{M}_\infty$;

(5) ρ is order continuous in Ɛ, i.e., g_n ∈ Ɛ, and $|g_n(w)|\downarrow 0$ implies $\rho(g_n)\downarrow 0$.

A set A ∈ ∑ is said to be ρ -null if $\rho(g\chi_A)=0$ for every $g\in$ Ɛ. A property $p(w)$ is said to hold ρ- almost everywhere $(\rho-a.e.)$ if the set $\{w\in\Omega:p(w)\,does\ not\ hold\}$ is ρ -null.

Definition 2.4[9] A regular function pseudomodular ρ is a regular convex function modular if $\rho(f)=0$ implies $f=0$ *a.e.* The class of all nonzero regular convex function modular defined on Ω will be denoted by $\Re$.

Definition 2.5 Let ρ be a nonzero regular convex function modular defined on Ω.

$$D_1(r,\varepsilon)=\{(f,g):f,g\in L_\rho,\ \rho(f)\le r,\ \rho(g)\le r,\rho(f-g)\ge\varepsilon r\}$$

Let

$$\delta_1(r,\varepsilon)=inf\{1-\frac{1}{r}\rho\left(\frac{f+g}{2}\right):(f,g)\in D_1(r,\varepsilon)\}$$

$if\ D_1(r,\varepsilon)\ne\phi,$

and $\delta_1(r,\varepsilon)=1$ if $D_1(r,\varepsilon)=\phi.$

Definition 2.6 A non-zero regular convex function modular ρ is said to satisfy (UC1) if every $r>0,\ \varepsilon>0,\ \delta_1(r,\varepsilon)>0.$ Note that for every $r>0,\ D_1(r,\varepsilon)\ne\phi$ for $\varepsilon>0$ small enough.

Definition 2.7 A non-zero regular convex function modular ρ is said to satisfy (UUC1) if for every $s\ge 0,\ \varepsilon>0,$ there exists $\eta_1(s,\varepsilon)>0$ depending only upon s and ε such that $\delta_1(r,\varepsilon)>\eta_1(s,\varepsilon)>0$ for any r > s.

Definition 2.8 [9] Let $\rho\in\Re$.

(1) A sequence $\{f_n\}$ is $\rho-$convergent to f, that is, $f_n\to f$ if $\rho(f_n-f)\to 0$ as $k\to\infty.$

(2) A sequence $\{f_n\}$ in is called $\rho-$Cauchy sequence if $\rho(f_n-f_m)\to 0$ as n, m → ∞.

(3) A set $B\subset L_\rho$ is called $\rho-$closed if for any sequence $\{f_n\}\subset B$, the convergence $f_n\to f$ as $n\to\infty$ implies that f belongs to B.

(4) A set $B\subset L_\rho$ is called $\rho-$bounded if its $\rho-$diameter is finite; the $\rho-$diameter of B is defined as $\delta_\rho(B)=\sup\{\rho(f-g):f,g\in B\}.$

(5) A set $B\subset L_\rho$ is called $\rho-$compact if for any sequence $\{f_n\}\subset B$, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and $f\in B$ such that $\rho(f_{n_k}-f)\to 0$ as $k\to\infty.$

(6) A set $B \subset L_\rho$ is called $\rho - a.e.$ closed if for any sequence $\{f_n\} \subset B$ which $\rho - a.e.$ converges $f_n \to f$ as $n \to \infty$ implies that f belongs to B.

(7) A set $B \subset L_\rho$ is called $\rho - a.e.$ compact if for any sequence $\{f_n\} \subset B$, there exists a subsequence $\{f_{n_k}\}$ and $f \in B$ such that $\rho(f_{n_k} - f) \to 0 a.e.$ as $k \to \infty$.

(8) Let $f \in L_\rho$ and $B \subset L_\rho$. The distance between f and B is defined as

$$d_\rho(f, B) = inf\{\rho(f - g) : g \in B\}.$$

Proposition 2.9[9]: Let $\rho \in \Re$.

(i) L_ρ is $\rho -$complete.

(ii) $\rho -$balls $B_\rho(f, r) = \{g \in L_\rho : \rho(f - g) \le r\}$ are $\rho -$closed.

(iii) If $\rho(\alpha f_n) \to 0$ for $\alpha > 0$ then there exists a subsequence $\{g_n\}$ of $\{f_n\}$ such that

$g_n \to 0\ \rho - a.e.$ as $n \to \infty$.

(iv) $\rho(f) \le lim\ inf_{n\to\infty} \rho(f_n)$ whenever $f_n \to f\ \rho - a.e.$ as $n \to \infty$. (Note: this property is equivalent to the Fatou property.)

(v) Consider the set

$L_\rho^0 = \{f \in L_\rho : \rho(f, .)\ is\ order\ continuous\}$ and,

$E_\rho = \{f \in L_\rho : \lambda f \in L_\rho^0\ for\ \text{any}\ \lambda > 0\}$. Then we have

$E_\rho \subset L_\rho^0 \subset L_\rho.$

Definition 2.10[9]: Let $\rho \in \Re$. Then ρ satisfies Δ_2 -property if $\rho(2f_n) \to 0$ whenever $\rho(f_n) \to 0$ as $n \to \infty$.

Proposition 2.11[9]: The following statements are equivalent:

(i) ρ satisfies Δ_2 – condition.

(ii) $\rho(f_n - f) \to 0$ if and only if $\rho(\lambda(f_n - f)) \to 0$, for every λ > 0 if and only if $\| f_n - f \|_\rho \to 0$ as $n \to 0$.

Definition 2.12[8] A set $C \subset L_\rho$ is called $\rho -$proximinal if for each $f \in L_\rho$, there exists an element g ∈ C such that

$$\rho(f - g) = d_\rho(f, C) = inf(\rho(f - h) : h \in C).$$

$P_\rho(C)$ denotes the family of nonempty ρ-bounded ρ-proximininal subset of C and $C_\rho(C)$ denotes the family of $\rho -$bounded $\rho -$closed subsets of C. Let $H_\rho(., .)$ be $\rho -$Hausdorff distance on $C_\rho(L_\rho)$, that is,

$$H_\rho(A, B) = max\{sup_{f\in A}\, dist \rho(f, B),\ sup_{g\in B}\, dist_\rho(g, A)\},$$

$A, B \in C_\rho(L_\rho)$.

Definition 2.13[8] A multivalued mapping $T : C \to C_\rho(L_\rho)$ is said to be $\rho -$Lipschitizian if there exists a number k ≥ 0 such that

$H_\rho(T(f), T(g)) \le k\rho(f - g)$ for all $f, g \in C$

(i) If k ≤ 1, then T is called $\rho -$nonexpansive

(ii) If k < 1, then T is called $\rho -$contractive

Lemma 2.14[2] Let ρ ò $\Re$ and satisfy (UUC1). Let $\{t_n\} \subset (0,1)$ be bounded away from both 0 and 1. If there exists R > 0 such that

$$lim_{n\to\infty} sup \rho(f_n) \le R, lim_{n\to\infty} sup \rho(g_n) \le R\ and$$
$$lim_{n\to\infty}(t_n f_n + (1 - t_n)\ g_n) = R,\ then$$
$$lim_{n\to\infty} \rho(f_n - g_n) = 0$$

The sequence $\{t_n\} \subset (0,1)$ is said to be bounded away from 0 if there exists a > 0 such that $t_n \ge a$ for all n ∈ N. Similarly the sequence $\{t_n\} \subset (0,1)$ is said to be bounded away from 1 if there exists b < 1 such that $t_n \le b$ for all n ∈ N.

Lemma 2.15[8] Let T: C→ $P_\rho(C)$ be a multivalued mapping and

$$P^T{}_\rho(f) = \{g \in T : \rho(f - g) = d_\rho(f, Tf)\}$$

Then the following are equivalent:

i) $f \in F_\rho(T)$, that is $f \in T(f)$;

ii) $P_\rho^T(f) = \{f\}, i.e., f = g$ for each $g \in P_\rho^T(f)$;

iii) $f \in F_\rho(P_\rho^T(f))$ that is, $f \in P_\rho^T(f)$. Further $F_\rho(T) = F(P_\rho^T)$ where $F(P_\rho^T)$ denotes the set of fixed points of P_ρ^T.

Lemma 2.16: Let ρ ò $\Re$ and satisfy Δ_2 –condition $\{f_n\}$ and $\{g_n\}$ be two sequences in L_ρ. Then

$lim_{n\to\infty} \rho(g_n) = 0\ implies$ and

$lim_{n\to\infty} sup \rho(f_n + g_n) = lim_{n\to\infty} sup \rho(f_n)$

$lim_{n\to\infty} \rho(g_n) = 0\ implies$

$lim_{n\to\infty} inf\ \rho(f_n + g_n) = lim_{n\to\infty} inf\ \rho(f_n).$

Lemma 2.17: Let ρ ò $\Re$ and A, B ∈ $P_\rho(L_\rho)$. For every f ∈ A, there exists g ∈ B such that

$$\rho\left(f-g\right)\le H_\rho\left(A,B\right)$$

Definition 2.18[5]: A family of mappings $T_i : C \to P_\rho(C)$ is said to satisfy condition (II) if there exists a nondecreasing function φ: [0, ∞) → [0, ∞) with φ(0) = 0, φ(r) > 0 for r ∈ (0, ∞) such that

$$d_\rho\left(f,\ T_i\left(f\right)\right)\ge\varphi\left(d_\rho\left(f,\ \cap_{i=1}^{m}F_\rho\left(T_i\right)\right)\right)$$

3. MAIN RESULTS

We prove some approximation theorems for Ishikwa iterative scheme which is more general than that of the Mann iterative scheme used by Safeer Husain Khan and Mujahid Abbas [8]. This iterative scheme is as follows:

Let C $\subset L_\rho$ be a non empty $\rho-$bounded, closed and convex set and $T : C \to P_\rho(C)$ be a multivalued mappings. Let $f_1 \in C$ and { f_n }⊂ C be defined by

$$g_n=\beta_n u_n+\ (1-\beta_n)\, f_n$$
$$f_{n+1}=\alpha_n v_n+\ (1-\alpha_n)\, f_n,\qquad n=1,2,\ldots\qquad(3.1)$$

where $u_n \in P_\rho^T(f_n), v_n\in P_\rho^T(g_n)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in (0,1) which are bounded away from both 0 and 1. Before proving our main results, we firstly prove the following Lemma:

Lemma 3.1: Let ρ ò $\Re$ and C ⊂ L_ρ be nonempty ρ-bounded and convex set. Suppose T : C → P_ρ (C) be a multivalued mappings such that P_ρ^T is ρ-nonexpansive mapping with F (T) ≠ ϕ. Then the $lim_{n\to\infty}\rho\left(f_n-p\right)$ exists for all p ∈ F.

Proof Let p ∈ F(T) be arbitrary. Then by Lemma 2.15, we have

$$P_\rho^T(p)=\{p\}$$

Now from eq. (3.1), we have

$$\rho\left(f_{n+1}-p\right)=\rho(\alpha_n v_n+(1-\alpha_n)\,f_n-\ p)$$
$$\le\alpha_n\rho\left(v_n-p\right)+(1-\alpha_n)\,\rho\left(f_n-p\right)$$
$$\le\alpha_n H_\rho(P_\rho^T\left(g_n\right)-P_\rho^T\left(p\right))+(1-\alpha_n)\,\rho\left(f_n-p\right)$$
$$\le\alpha_n\rho\left(g_n-p\right)+(1-\alpha_n)\,\rho\left(f_n-p\right)\qquad(3.2)$$

Again from eq. (3.1), we get that

$$\rho\left(g_n-p\right)=\rho(\beta_n u_n+(1-\beta_n)\,f_n-p)$$
$$\le\beta_n\,\rho\left(u_n-p\right)+(1-\beta_n)\,\rho(f_n-p)$$
$$\le\beta_n\,\rho(P_\rho^T\left(f_n\right)-P_\rho^T\left(p\right))\ +\ (1-\beta_n)\,\rho\left(f_n-p\right)\qquad(3.3)$$
$$\le\beta_n\,\rho\left(f_n-p\right)+(1-\beta_n)\,\rho\left(f_n-p\right)$$
$$=\rho\left(f_n-p\right)$$

Then from eq, (3.2), eq. (3.3), we obtained that

$$\rho\left(f_{n+1}-p\right)\le\rho\left(f_n-p\right)$$

Therefore, the sequence { $\rho\left(f_n-p\right)$ } is decreasing. Hence $lim_{n\to\infty}\rho\left(f_n-p\right)$ exists for all p ∈ $F_\rho(T)$.

Theorem 3.2: Let ρ ò $\Re$ satisfy (UUC1) and C ⊂ L_ρ be nonempty $\rho-$bounded, closed and convex set. Suppose T : C → P_ρ (C) be a multivalued mappings such that P_ρ^T is ρ-nonexpansive mappings with $F_\rho(T)$ ≠ ϕ.

Let f_1 ∈C and { f_n } be given by (3.1). Then

$$lim_{n\to\infty}\rho\left(f_n-u_n\right)=0$$

Proof By Lemma (3.1), $lim_{n\to\infty}\rho\left(f_n-p\right)$ exists for all p ∈ $F_\rho(T)$. Let

$$\lim_{n\to\infty}\rho(f_n-p)=R\quad(3.4)$$

From eq. (3.3) and eq. (3.4), we get

$$\lim_{n\to\infty}\sup\rho(g_n-p)\le R\quad(3.5)$$

This implies

$$\rho\left(v_n-\ p\right)\le\rho(P_\rho^T\left(g_n\right)-P_\rho^T\left(p\right))$$
$$\le\rho\left(g_n-\ p\right)\le\rho\left(f_n-\ p\right)\ \text{[using eq. (3.3)]}$$

which implies that

$$lim_{n\to\infty}\sup\rho\left(v_n-p\right)\le lim_{n\to\infty}\sup\rho\left(f_n-p\right)$$

or $lim_{n\to\infty}\sup\rho\left(v_n-p\right)\le R$ (3.6)

Similarly

$$\rho\left(u_n-\ p\right)\le\rho(P_\rho^T\left(f_n\right)-P_\rho^T\left(p\right))$$
$$\le\rho\left(f_n-\ p\right)\ \text{[using eq. (3.3)]}$$

which implies that

$$lim_{n\to\infty}\sup\rho\left(u_n-p\right)\le lim_{n\to\infty}\sup\rho\left(f_n-p\right)$$

or $lim_{n\to\infty}\sup\rho\left(u_n-p\right)\le R$ (3.7)

Since the sequence $\{\alpha_n\}$ ⊂ (0, 1) is bounded away from 0 and 1, there exists α ∈ (0, 1) such that

$$lim_{n\to\infty}\alpha_n=\alpha$$

Now,

$$\rho\left(f_{n+1}-p\right)=\rho(\alpha_n v_n+(1-\alpha_n)\,f_n-p)$$

$= \rho(\alpha_n (v_n - p) + (1-\alpha_n)(f_n - p))$

$\le \alpha_n \rho (v_n - p) + (1-\alpha_n)\ \rho(f_n - p)$

$lim_{n\to\infty} inf\ \ \rho(f_{n+1} - p)$

$\le lim_{n\to\infty} inf\ \ (\alpha_n \rho (v_n -\ p) + (1-\alpha_n)\ \rho(\ f_n -\ p))$

$\le\ lim_{n\to\infty} inf\ \ (\alpha_n \rho (v_n - p)) + lim_{n\to\infty} inf\ \ (1-\alpha_n)\ \rho(f_n - p)$

or $R \le \alpha lim_{n\to\infty} inf\ \ \rho(v_n - p) + (1-\alpha)\ R$

which implies that

$R \le lim_{n\to\infty} inf\ \ \rho(v_n - p)$ (3.8)

From eq. (3.6) and eq. (3.8), we get

$lim_{n\to\infty} \rho(v_n - p) = R$

Since $v_n \in P_\rho^T(g_n)$, then

$\rho(v_n - p) \le \rho(g_n - p)$ which implies that

$lim_{n\to\infty} inf\ \rho(g_n - p) \ge R$ (3.9)

Then from eq. (3.5) and (3.9)

$lim_{n\to\infty} \rho(g_n - p) = R$ (3.10) Since the sequence { β_n }⊂ (0, 1) is bounded away from 0 and 1, there exists β ∈ (0, 1) such that

$lim_{n\to\infty} \beta_n = \beta$

Now,

$\rho(g_n - p) = \rho(\beta_n u_n + (1-\beta_n)\ f_n - p)$

$\le \beta_n \rho(u_n - p) +\ \ (1-\beta_n)\ \rho(f_n - p)$

which implies that

$lim_{n\to\infty} inf\ \ \rho(g_n - p)$

$\le lim_{n\to\infty} inf\ \beta_n \rho(u_n - p) + lim_{n\to\infty} inf\ \ (1-\beta_n)\ \rho(f_n - p)$

$R \le \beta lim_{n\to\infty} inf\ \ \rho(u_n - p) + (1-\beta)\ R$

Or $lim_{n\to\infty} inf\ \ \rho(u_n - p) \ge R$ (3.11)

Then from eq. (3.7) and eq. (3.11), we have

$lim_{n\to\infty} \rho(u_n -\ p) = R$ (3.13)

Since $\lim_{n\to\infty} \rho(g_n - p) = R$

This implies

$lim_{n\to\infty} \rho(\beta_n u_n + (1-\beta_n)\ f_n - p) = R$

or $lim_{n\to\infty} \rho(\beta_n (u_n - p) + (1-\beta_n)(f_n - p)) = R$ (3.12)

Then from eq. (3.4), eq. (3.7), eq. (3.12) and Lemma (2.14), we obtain that

$lim_{n\to\infty} \rho(f_n - u_n) = 0$ or

$lim_{n\to\infty} dis_\rho \rho(f_n, P_\rho^T(f_n)) = 0$

Theorem 3.3: Let $\rho \in \Re$ satisfy (UUC1) and C ⊂ L_ρ be nonempty $\rho-$bounded and convex set. Let T : C → P_ρ(C) be a multivalued mapping such that P_ρ^T is $\rho-$nonexpansive mapping with $F_\rho(T) \ne$ ϕ. Let $f_1 \in$ C and { f_n } be given by (3.1). Then the sequence { f_n } converges to a fixed point of T.

Proof: Using the compactness of C, there must be a subsequence { f_{n_k} } of { f_n } and f ∈ C such that $\rho(f_{n_k} - f) \to 0$ as $k \to \infty$. We will show that f is a fixed point of T, i.e., $f \in F_\rho(T)$.

Let $g \in P_\rho^T(f)$ be arbitrary. Then by Lemma (2.16), $g_k \in P_\rho^T(f_{n_k})$ such that

$\rho(g_k - g) \le H_\rho(P_\rho^T(f_{n_k}), P_\rho^T(f))$

We

$$\rho\left(\frac{f-g}{3}\right) = \rho\left(\frac{f-f_{n_k}}{3} + \frac{f_{n_k}-g_k}{3} + \frac{g_k - g}{3}\right)$$

$$\le \frac{1}{3}\rho(f - f_{n_k}) + \frac{1}{3}\rho(f_{n_k} - g_k) + \frac{1}{3}\rho(g_k - g)$$

$$\le \rho(f - f_{n_k}) + dis_\rho(f_{n_k} - P_\rho^T(f_{n_k})) + \rho(g_k - g)$$

$$\le \rho(f - f_{n_k}) + dis_\rho(f_{n_k} - P_\rho^T(f_{n_k})) + H_\rho(P_\rho^T(f_{n_k}), P_\rho^T(f))$$

$$\le \rho(f - f_{n_k}) + d_\rho(f_{n_k} - P_\rho^{T_1}(f_{n_k})) + \rho(f - f_{n_k}) \to 0$$

as k→∞.

Hence $f = g$ a.e. Since $g \in P_\rho^T(f)$ was arbitrary, we have $P_\rho^T(f)$ = { f }. Thus by using Lemma (2.15), $f \in T(f)$) and thus f ∈ F_ρ(T).

This completes the proof.

Theorem 3.4: Let $\rho \in \Re$ satisfy (UUC1) and C ⊂ L_ρ be nonempty ρ-bounded and convex set. Let T : C → P_ρ(C) be a multivalued mapping such that P_ρ^T is $\rho-$nonexpansive mappings with $F_\rho(T) \ne \phi$. Let f_1∈ C and { f_n } be given by eq. (3.1). Suppose T satisfy condtion (II). Then the sequence { f_n } converges a fixed point of T.

Proof: By using Lemma 3.1, we obtained that $lim_{n\to\infty} \rho(f_n - p)$ exists for all p ∈ $F_\rho(T) \ne \phi$.

If $lim_{n\to\infty}\rho\left(f_n - p\right)$ = 0, then nothing to do. Assume that $lim_{n\to\infty}\rho\left(f_n - p\right)$ = R > 0. By same Lemma, we have $\rho\left(f_{n+1} - p\right) \le \rho\left(f_n - p\right)$, for all p ∈ $F_\rho(T) \ne \phi$.
This implies that
$$dis_\rho\left(f_{n+1},\ F_\rho(T)\right) \le dis_\rho\left(f_n,\ F_\rho(T)\right).$$
So that $lim_{n\to\infty}dis_\rho\left(f_n, F_\rho(T)\right)$ exists. By Theorem 3.2 and condition (II) either
$0 = lim_{n\to\infty}d_\rho\left(f_n,\ T\left(f_n\right)\right) \ge lim_{n\to\infty}\varphi\left(dis_\rho\left(f_n,\ F\right)\right)$ In all cases, $lim_{n\to\infty}\varphi\left(dis_\rho\left(f_n,\ F_\rho(T)\right)\right) = 0$. Since φ is increasing and $\varphi(0) = 0$, it must be the case that $lim_{n\to\infty}dis_\rho\left(f_n,\ F_\rho(T)\right) = 0.$
Let ε > 0 be arbitrary. Then there exists an integer m_0 ∈ N such tha $dis_\rho\left(f_n,\ F_\rho(T)\right) < \frac{\varepsilon}{2}$, for all $n \ge m_0$.

Particularly, $inf\{\rho\left(f_{m_0} - p\right):\ p \in F\} < \frac{\varepsilon}{2}$. Thus, there exists a p_0 ∈ F such that
$$\rho\left(f_{m_0} - p_0\right) < \varepsilon \quad (3.14)$$
Now for $m, n \ge m_0$, we have
$$\rho\left(\frac{f_n - f_m}{2}\right) \le \left(\frac{1}{2}\right)\rho\left(f_m - p_0\right) + \left(\frac{1}{2}\right)\rho\left(f_n - p_0\right)$$
$$\le \left(\frac{1}{2}\right)\rho\left(f_{m_0} - p_0\right) + \left(\frac{1}{2}\right)\rho\left(f_{m_0} - p_0\right)$$
$$< \varepsilon$$
Since ρ satisfies $\Delta_2 -$condition, by Proposition 2.11, we get { f_n } is a $\rho-$Cauchy sequence in C. As L_ρ is complete and C is ρ-closed, then there must exist an f ∈ C such that $\rho\left(f_n - f\right) \to 0.$ By Theorem 3.3 the required result is proved.

4. CONFLICT OF INTERESTS:

The authors state that there is no conict of interests concerning the publication of this paper.

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